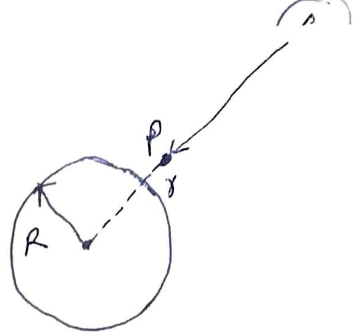
Sol^n From Gauss's Law, the field outside is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r},$$



$q \rightarrow$ total charge on the sphere

The field inside is zero. For the points outside the sphere $(r > R)$

$$V(r) = -\int_0^r E \cdot dl = \frac{-1}{4\pi\varepsilon_0}\int_\infty^r \frac{q}{r'^2}\,dr'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\Big|_\infty^r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Potential inside the sphere $(r < R)$, we must break the integral into two sections

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_\infty^R \frac{q}{r'^2}\,dr' - \int_R^r (0)\,dr'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\Big|_\infty^R + 0$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

Potential is not zero inside the shell, even though field is.

$V \rightarrow$ constant $\rightarrow \nabla V = 0$

The potential inside the sphere is sensitive to what's going on outside the sphere as well.

Poisson's Equation and Laplace's Equation

$$E = -\nabla V$$

What do the fundamental eq^{n}s for E

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla \times E = 0$$

look like, in terms of V?

$$\nabla \cdot E = \nabla \cdot (-\nabla V)$$
$$= -\nabla^2 V$$
$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}} \rightarrow \text{Poisson's eq}^n$$

In regions where there is no charge, $\rho = 0$

$\boxed{\nabla^2 V = 0}$ Laplace eq^n

$$\nabla \times E = \nabla \times (-\nabla V)$$
must equal zero

Curl of gradient is always zero

↳ E could be expressed as the gradient of a scalar, $\nabla \times E = 0$ permits $E = -\nabla V$

in return $E = -\nabla V$ guarantees $\nabla \times E = 0$

It takes only one differential eq^n (Poisson's eq^n) to determine V. because V is scalar

E → need two, the divergence and curl.

Electrostatic Boundary Conditions

Source charge distribution ρ → given

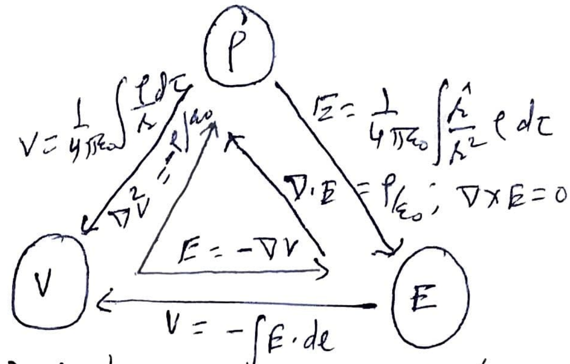
↓

want to find the electric field E it produces

Unless symmetry of the problem admits a sol^n by Gauss's Law, it is generally to your advantage to calculate the potential first as an intermediate step.

Three fundamental quantities of electrostatics;

ρ, E and [illegible]



We began with just two experimental observations

(1) The principle of superposition

(2) Coulomb's Law

From these, all else followed.

Electric field always undergoes a discontinuity when you cross a surface charge σ.

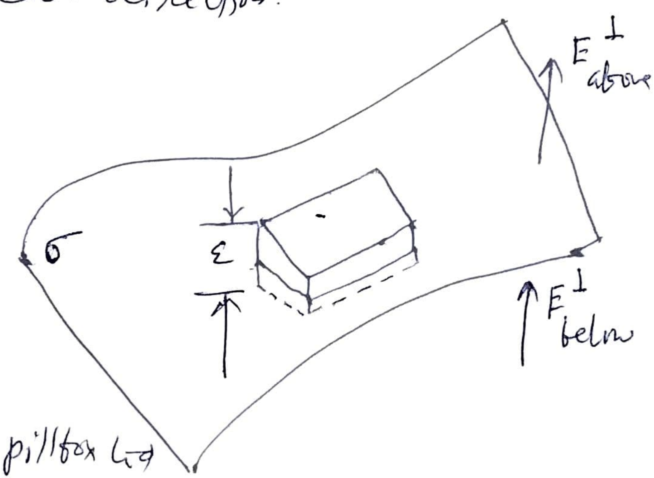
Amount by which E changes at such a boundary?

We draw a wafer-thin Gaussian pillbox, extending just barely over the edge in each direction.

Gauss's Law states that

$$\oint_S E \cdot da = \frac{1}{\varepsilon_0} Q_{enc}$$

$$= \frac{1}{\varepsilon_0} \sigma A$$



$A \to$ area of the pillbox lid

Sides of the pillbox contribute nothing to the flux, in the limit as the thickness ε goes to zero

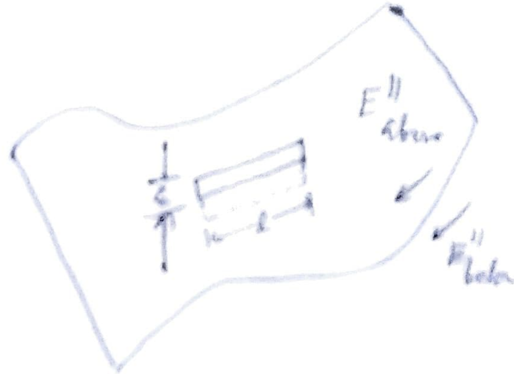
$$\Rightarrow \quad E^\perp_{above} - E^\perp_{below} = \frac{\sigma}{\varepsilon_0}$$

Normal component of E is discontinuous by an amount $\frac{\sigma}{\varepsilon_0}$ at any boundary. In particular, where there is no surface charge, $E^\perp$ is continuous $\to$ surface of uniformly charged solid sphere.

The tangential component of E, by contrast, is always continuous

if we apply $\oint E \cdot dl = 0$

to the thin rectangular loop



The ends give nothing (as $\epsilon \to 0$) and the sides give $(E^{\parallel}_{above} l - E^{\parallel}_{below} l)$, so

$$E^{\parallel}_{above} = E^{\parallel}_{below}$$

$E^{\parallel} \to$ component of E parallel to the surface

The boundary conditions on E can be combined into a single formula

$$E_{above} - E_{below} = \frac{\sigma}{\epsilon_0} \hat{n}$$

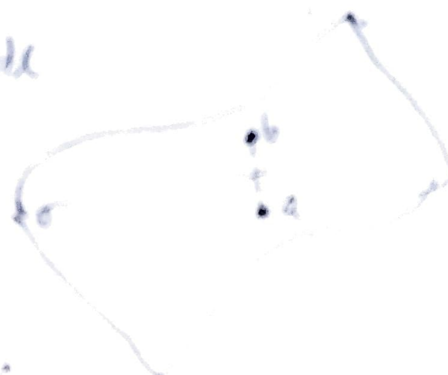
$\hat{n}$ ↳ unit vector perpendicular to the surface

Pointing from below to above

The potential → is continuous across any boundary since

$$V_{above} - V_{below} = - \int_a^b E \cdot dl$$

As the path length shrinks to zero, so too does the integral



$$V_{above} = V_{below}$$

The gradient of V inherits the ...

[illegible]

Laplace's Equations

Primary task of electrostatics → To find the electric field of a given stationary charge distribution.

Coulomb's Law $\xrightarrow{}$ Can be difficult to calculate

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,\rho(r')\,d\tau' \quad \text{—(1)}$$

↓

exploiting symmetry and Gauss's Law.

Best strategy is to first calculate the potential V,

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\,\rho(r')\,d\tau' \quad \text{—(2)}$$

↓

even this integral is often too tough to handle analytically.

Moreover, in problems involving conductors ρ itself may not be known in advance: Since charge is free to move around

↳ Only thing could be controlled directly is the total charge of each conductor

It is useful to recast the problem in differential form

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho \quad \text{—(3)}$$

which, together boundary conditions is equivalent to eq^n(2).

Very often, we are interested in finding the potential in a region where $\rho = 0$

$$\nabla^2 V = 0 \quad \text{—(4)}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{—(5)}$$

Laplace's Equation in One Dimension

Suppose V depends on one variable, x.

$$\frac{d^2V}{dx^2} = 0$$

The general sol^n is $\underline{V(x) = mx + b}$ — (6)

eq^n for a straight line.

Two undetermined constants (m and b), appropriate for a second-order differential eq^n.

Fixed → by boundary conditions of that problem.

Two features of this result:

1. $V(x)$ is the average of $V(x+a)$ and $V(x-a)$ for any a:

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$$

Laplace's eq^n → a kind of averaging instruction
→ it tells you to assign to the point x, the average of the values to the left and to the right of x.

2. Laplace's eq^n tolerates no local maxima or minima → extreme values of V must occur at the end points.

Laplace's eq^n in Two Dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

no longer ordinary differential eq^n
↳ partial differential eq^n

One can't write down a "general sol^n". However, it is possible to deduce certain properties common to all solutions.

Uniqueness Theorem :- Several methods for solving a given problem → analytical, graphical, numerical, experimental etc.

Solving Laplace's eqⁿ in different ways — different solⁿ? ✗

If a solⁿ of Laplace's eqⁿ satisfies a given set of boundary conditions → is this the only possible solⁿ → yes

↓

Solⁿ is unique

Any solⁿ of Laplace's eqⁿ that satisfies the same boundary conditions must be the only solⁿ regardless of the method used ⟶ uniqueness theorem

Assume that there are two sol'ⁿs V_1 and V_2 of Laplace's eqⁿ both of which satisfy the prescribed boundary conditions.

$$\nabla^2 V_1 = 0, \quad \nabla^2 V_2 = 0 \quad \text{--- (1)}$$

$$V_1 = V_2 \text{ on the boundary} \quad \text{--- (2)}$$

We consider

$$V_d = V_2 - V_1 \quad \text{--- (3)}$$

which obeys

$$\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0 \quad \text{--- (4)}$$

$$V_d = 0 \text{ on the boundary} \quad \text{--- (5)}$$

From divergence theorem

$$\int_v \nabla \cdot A \, dv = \oint_s A \cdot ds \quad \text{--- (6)}$$

S → surface surrounding volume v and is the boundary of the original problem

Let $A = V_d \nabla V_d$ and use vector identity

$$\nabla \cdot A = \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d$$

but $\nabla^2 V_d = 0$ so

$$\nabla \cdot A = \nabla V_d \cdot \nabla V_d \quad \text{---(7)}$$

$$\Rightarrow \int_V \nabla V_d \, \nabla V_d \, dv = \oint_S V_d \nabla V_d \cdot ds \quad \text{---(8)}$$

from eqⁿ (1), (2) & (4), (5) R.H.S of eqⁿ (8) vanishes

$$\Rightarrow \int_V |\nabla V_d|^2 \, dv = 0$$

Since integrand is always positive.

$$\nabla V_d = 0 \quad \text{---(9)}$$

or $V_d = V_2 - V_1$ = constant everywhere in V ---(10)

eqⁿ (10) must be consistent with eqⁿ (2)

Hence $V_d = 0$ or $V_1 = V_2$ everywhere,

$\Rightarrow$ V_1 and V_2 can not be different solⁿ's of the same problem.

$\downarrow$

Similar theorem applies to Poisson's eqⁿ also